12-3 CHEBYSHEV FILTERS

An alternative way of approximating the ideal characteristics of Fig. 12-3 is to obtain an *equal-ripple* error in the range $0 \leq \omega \leq \omega_p$. It is convenient to introduce immediately frequency normalization, with

$$\omega_0 = 2\pi f_p \tag{12-38}$$

as the frequency unit; i.e., we are normalizing to the passband limit. The desired response is then that shown in Fig. 12-11. The oscillatory response in the passband immediately suggests a squared and horizontally compressed trigonometric function. Hence, we attempt to find the solution for $|K|^2$ in the form†

$$|K|^2 = k_p^2 \cos^2 nu(\Omega) \tag{12-39}$$

where $u(\Omega)$ is some function of Ω. If we choose

$$u(\Omega) = \cos^{-1} \Omega \tag{12-40}$$

the following conclusions can be drawn:

1. $|K|^2$ is a polynomial in Ω^2, since

$$\cos nu = \text{Re}\,(e^{ju})^n = \text{Re}\,(\cos u + j \sin u)^n$$
$$= \cos^n u - \binom{n}{2} \cos^{n-2} u\,(1 - \cos^2 u) + \binom{n}{4} \cos^{n-4} u\,(1 - \cos^2 u)^2 - + \cdots \tag{12-41}$$

† Of course, this solution can also be found in a less heuristic way. See, for example, N. Balabanian, "Network Synthesis," Prentice-Hall, Englewood Cliffs, N.J., 1958.

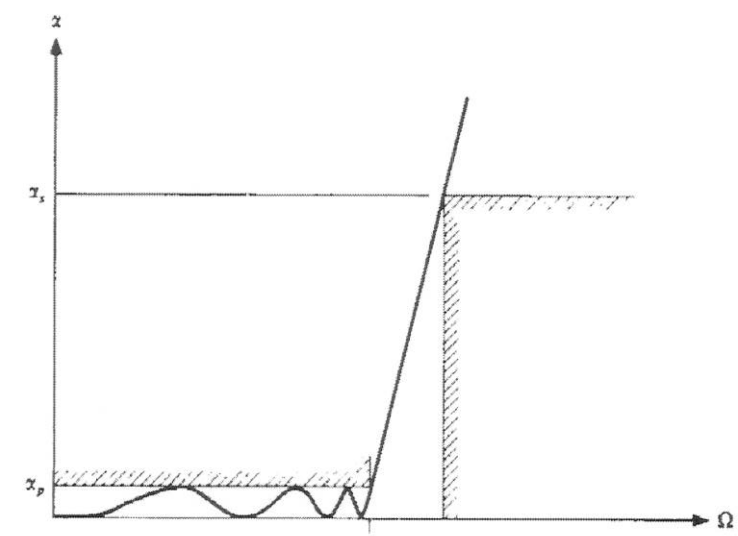

Figure 12-11 Chebyshev passband loss response.

where $\binom{n}{k} = n!/[k!(n-k)!]$. From Eq. (12-41) it is obvious that for n even (odd), $\cos nu$ is a pure even (odd) polynomial in $\cos u$. From this fact and Eqs. (12-39) and (12-40) our statement follows.

2. $|K|^2$ oscillates between 0 and k_p^2 for $-1 \leq \Omega \leq +1$. This is true, since by Eq. (12-40) in this range u is real and as Ω grows from -1 to $+1$, u can be considered to grow from $-\pi$ to 0. (In fact, $\cos^{-1} \Omega$ is multivalued, and hence this choice is arbitrary.) Hence, $|K|^2$ oscillates between zero and k_p^2, taking on the value 0 a total of n times and the value k_p^2 a total of $n+1$ times between $-\Omega_p = -1$ and $\Omega_p = +1$.
3. For values of Ω greater than 1, $|K|^2$ tends monotonically to infinity. This can be seen from the relation

$$\begin{aligned}\cos nu &= \tfrac{1}{2}(e^{jnu} + e^{-jnu}) \\ &= \tfrac{1}{2}[(\cos u + \sqrt{\cos^2 u - 1})^n + (\cos u + \sqrt{\cos^2 u - 1})^{-n}] \\ &= \tfrac{1}{2}[(\Omega + \sqrt{\Omega^2 - 1})^n + (\Omega + \sqrt{\Omega^2 - 1})^{-n}] \qquad (12\text{-}42)\end{aligned}$$

For $\Omega \to \infty$, $\cos nu \to 2^{n-1}\Omega^n$, and hence $|K|^2 \to k_p^2 2^{2n-2}\Omega^{2n}$.

Due to the described properties, the filter defined by Eqs. (12-39) and (12-40) does indeed have the behavior shown in Fig. 12-11 (which illustrates the $n = 7$ case). The maximum passband loss α_p and k_p are related, from Eq. (12-1), by

$$\alpha_p = 10 \log (1 + k_p^2) \qquad (12\text{-}43)$$

Filters which have $|K|^2$ given by (12-39) and (12-40) are called *Chebyshev filters* after the mathematician first analyzing the properties of the polynomials $\cos (n \cos^{-1} x)$, called *Chebyshev polynomials.*

Let us now compare the stopband responses of a Butterworth and a Chebyshev filter with the same passband limit, say $\Omega_p = 1$, and the same maximum passband loss α_p. In the stopband, typically $\alpha > 30$ dB, and hence by (12-1) $|K|^2 \gg 1$; therefore

$$\alpha \approx 20 \log |K| \qquad (12\text{-}44)$$

Then, for the loss α_B of the Butterworth filter, using Eq. (12-34), we get

$$\alpha_B \approx 10 \log k_p^2\Omega^{2n} = 20 \log k_p + 20n \log \Omega \qquad (12\text{-}45)$$

while for the loss α_{Ch} of the Chebyshev filter of the same degree n,

$$\alpha_{Ch} \approx 20 \log k_p + (10)(2n - 2) \log 2 + 20n \log \Omega \qquad (12\text{-}46)$$

Hence, at the same stopband frequency Ω,

$$\alpha_{Ch} \approx \alpha_B + 6.02(n - 1) \qquad (12\text{-}47)$$

For even moderate degrees, the additional stopband loss $6.02(n-1)$ dB of the Chebyshev filter is significant. For example, for $n = 5$, the added loss is over 24 dB. This illustrates that the Chebyshev filter is significantly more efficient in its loss characteristics than the Butterworth filter.

Since $|K|^2$ is a polynomial function of Ω^2, as we have already seen from

Eqs. (12-39) to (12-41), all poles of $|K|^2$, that is, all loss poles, lie at $\Omega \to \infty$. The reflection zeros, by Eq. (12-39), are located at values of u satisfying

$$nu_k^{(r)} = \frac{2k-1}{2}\pi \qquad k = 1, 2, \ldots, n \tag{12-48}$$

or, using Eq. (12-40),

$$\Omega_k^{(r)} = \cos u_k^{(r)} = \cos \frac{2k-1}{2n}\pi \qquad k = 1, 2, \ldots, n \tag{12-49}$$

The natural modes S_k and their mirror images can be found by extending Eqs. (12-39) and (12-40), which are valid on the $j\Omega$ axis only. Hence, replacing Ω by S/j, we get

$$|K|^2 + 1 = k_p^2 \cos^2 nu_k + 1 = 0 \qquad S_k = j \cos u_k \qquad k = 1, 2, \ldots, 2n \tag{12-50}$$

We have to select the S_k in the LHP as the natural modes.

Anticipating complex solutions for both u_k and S_k, we have by analytic continuation

$$u_k = v_k + jw_k \qquad S_k = \Sigma_k + j\Omega_k \tag{12-51}$$

where v_k, w_k, Σ_k, and Ω_k are all real. Then, using the familiar identity

$$\cos(x + jy) = \cos x \cos jy - \sin x \sin jy = \cos x \cosh y - j \sin x \sinh y \tag{12-52}$$

by the first relation in Eq. (12-50) we have

$$\cos nu_k = \cos nv_k \cosh nw_k - j \sin nv_k \sinh nw_k = \pm \frac{j}{k_p} \tag{12-53}$$

Equating real and imaginary parts on both sides, we get

$$\cos nv_k \cosh nw_k = 0 \qquad \sin nv_k \sinh nw_k = \pm \frac{1}{k_p} \tag{12-54}$$

Since $\cosh nw_k > 0$, we can write

$$nv_k = \pm \frac{2k-1}{2}\pi \qquad k = 1, 2, \ldots \tag{12-55}$$

and hence $\sin nv_k = \pm 1$ and

$$\sinh nw_k = \pm \frac{1}{k_p} \tag{12-56}$$

$$w_k = \pm \frac{1}{n} \sinh^{-1} \frac{1}{k_p} = \pm \frac{1}{n} \ln \left(\frac{1}{k_p} + \sqrt{\frac{1}{k_p^2} + 1} \right)$$

At this stage, v_k and w_k are known, from Eqs. (12-55) and (12-56). Substituting into Eq. (12-50) gives

$$S_k = \Sigma_k + j\Omega_k = j \cos u_k = j \cos (v_k + jw_k)$$
$$\Sigma_k + j\Omega_k = j \cos v_k \cosh w_k + \sin v_k \sinh w_k \qquad (12\text{-}57)$$

Hence,
$$\Sigma_k = \sin v_k \sinh w_k \qquad \Omega_k = \cos v_k \cosh w_k \qquad (12\text{-}58)$$

or, from Eqs. (12-55) and (12-56),

$$\Sigma_k = -\sin \left(\frac{2k-1}{n}\frac{\pi}{2}\right)\frac{1}{2}(a^{1/n} - a^{-1/n})$$
$$\Omega_k = \cos \left(\frac{2k-1}{n}\frac{\pi}{2}\right)\frac{1}{2}(a^{1/n} + a^{-1/n}) \qquad (12\text{-}59)$$

In Eq. (12-59), $k = 1, 2, \ldots, n$, and

$$a \triangleq \frac{1}{k_p} + \sqrt{\frac{1}{k_p^2} + 1} \qquad (12\text{-}60)$$

The locus of the S_k in the S plane can be found by solving the two equations in (12-59) for $\sin [(2k-1)\pi/2n]$ and $\cos [(2k-1)\pi/2n]$, respectively, and then squaring and adding the two relations. This gives

$$\frac{\Sigma_k^2}{[(a^{1/n} - a^{-1/n})/2]^2} + \frac{\Omega_k^2}{[(a^{1/n} + a^{-1/n})/2]^2} = \sin^2 \left(\frac{2k-1}{n}\frac{\pi}{2}\right) + \cos^2 \left(\frac{2k-1}{n}\frac{\pi}{2}\right)$$
$$= 1 \qquad (12\text{-}61)$$

Hence, the locus is an *ellipse*, with half axes $(a^{1/n} + a^{-1/n})/2$ and $(a^{1/n} - a^{-1/n})/2$. The natural modes for the $n = 4$ case are illustrated in Fig. 12-12.

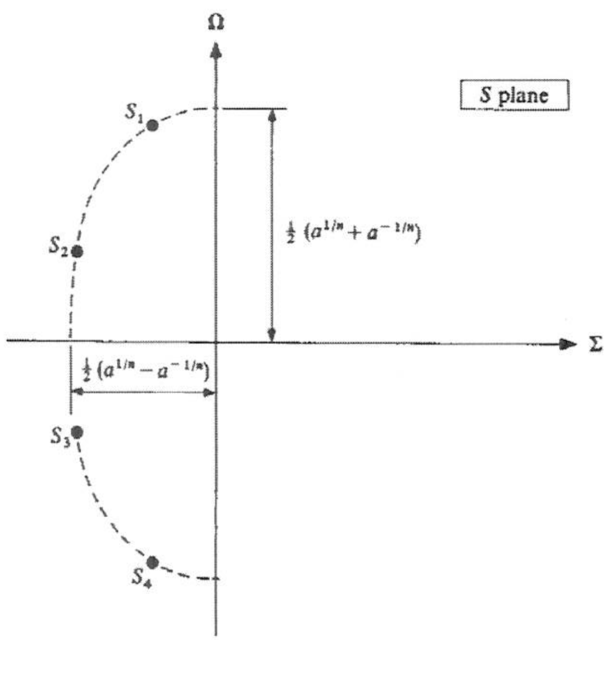

Figure 12-12 Natural modes of a Chebyshev filter for $n = 4$.

Next, the degree n necessary to satisfy a given set of specifications will be derived. For requirements of the form shown in Eq. (12-20), from Eqs. (12-39) and (12-40), we find that the passband requirement is always satisfied if k_p is obtained from (12-43) and if the normalization

$$\Omega = \frac{\omega}{\omega_0} = \frac{2\pi f}{2\pi f_p} = \frac{f}{f_p} \tag{12-62}$$

is used. To satisfy the stopband specification (Fig. 12-11), we must have

$$\alpha(\Omega_s) = 10 \log [|K(j\Omega_s)|^2 + 1] \geq \alpha_s \tag{12-63}$$

By Eqs. (12-39) and (12-40), therefore,

$$k_p^2 \cos^2 nu(\Omega_s) \geq 10^{\alpha_s/10} - 1 \qquad u(\Omega_s) = \cos^{-1} \Omega_s \tag{12-64}$$

Now since $\Omega_s > 1$, $u(\Omega_s)$ is imaginary, i.e.,

$$u(\Omega_s) = jw_s \tag{12-65}$$

but

$$w_s = -ju(\Omega_s) = -j \cos^{-1} \Omega_s = \cosh^{-1} \Omega_s \tag{12-66}$$

is real. By Eq. (12-64),

$$k_p \cos njw_s = k_p \cosh nw_s \geq \sqrt{10^{\alpha_s/10} - 1} \tag{12-67}$$

Hence, from Eqs. (12-66) and (12-67),

$$nw_s = n \cosh^{-1} \Omega_s \geq \cosh^{-1} \frac{\sqrt{10^{\alpha_s/10} - 1}}{k_p} \tag{12-68}$$

Next, we can express k_p from (12-43)

$$k_p = \sqrt{10^{\alpha_p/10} - 1} \tag{12-69}$$

and Ω_s from (12-23)

$$\Omega_s = \frac{\Omega_p}{k} = \frac{1}{k} \tag{12-70}$$

Substituting into (12-68), and using (12-24), we obtain

$$n \geq \frac{\cosh^{-1} \sqrt{(10^{\alpha_s/10} - 1)/(10^{\alpha_p/10} - 1)}}{\cosh^{-1} \Omega_s} = \frac{\cosh^{-1} (1/k_1)}{\cosh^{-1} (1/k)} \tag{12-71}$$

which is the desired result. It is of some interest to note the similarity of Eqs. (12-25) and (12-71). In both, n can be expressed exclusively in terms of k and k_1.

Example 12-4 Design a Chebyshev low-pass filter for the specifications

$\alpha_p = 1$ dB $\quad f_p = 1.8$ MHz $\quad \alpha_s = 50$ dB $\quad f_s = 7$ MHz $\quad R_G = R_L = 50\ \Omega$

satisfied earlier by a Butterworth filter.

As found before, $k = 0.257143$ and $k_1 = 1.60912 \times 10^{-3}$. Hence, from (12-71),

$$n \geq \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} \approx 3.5025$$

Hence, $n = 4$ can be chosen, one lower than for the Butterworth filter. From (12-41) and (12-40) then

$$\cos 4u = \cos^4 u - \binom{4}{2} \cos^2 u(1 - \cos^2 u) + \binom{4}{4}(1 - \cos^2 u)^2 = 8\Omega^4 - 8\Omega^2 + 1$$

Hence, by (12-39),

$$|K|^2 = k_p^2(8\Omega^4 - 8\Omega^2 + 1)^2$$

where now from (12-69)

$$k_p = \sqrt{10^{\alpha_p/10} - 1} = \sqrt{10^{0.1} - 1} \approx 0.508847$$

Hence, we can choose

$$K(S) = \pm k_p(8S^4 + 8S^2 + 1) \approx \pm(4.070776S^4 + 4.070776S^2 + 0.508847)$$

The natural modes can be obtained, say from (12-58), where now, by (12-55),

$$v_k = \frac{\pm(k - \frac{1}{2})\pi}{4} \qquad k = 1, 2, 3, 4$$

and, by (12-56),

$$w_k = \pm\tfrac{1}{4} \sinh^{-1} \frac{1}{k_p} = \pm 0.356994$$

Hence, for the first natural mode

$$\Sigma_1 = -\sin \frac{\pi}{8} \sinh 0.356994 \approx -0.139536 \qquad \Omega_1 = \cos \frac{\pi}{8} \cosh 0.356994 \approx 0.983379$$

Proceeding this way, we can find all $S_k = \Sigma_k + j\Omega_k$. We obtain $S_4 = S_1^*$ and $S_2 = S_3^* = -0.33687 + j0.407329$. Then

$$H(S) = C \prod_{i=1}^{4} (S - S_i)$$

Here the constant factor C is the coefficient of S^4 in $H(S)$. By the Feldtkeller equation, the coefficients of S^4 in $K(S)$ and $H(S)$ must have the same absolute values. Hence $C = \pm 4.070776$. The overall function is thus found to be

$$\begin{aligned} H(S) &= \pm 4.070776(S^2 - 2\Sigma_1 S + \Sigma_1^2 + \Omega_1^2)(S^2 - 2\Sigma_2 S + \Sigma_2^2 + \Omega_2^2) \\ &= \pm(a_4 S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0) \end{aligned}$$

where $a_4 = 4.070776 \qquad a_3 = 3.878684 \qquad a_2 = 5.9186$

$$a_1 = 3.02304 \qquad a_0 = 1.12202$$

Hence, when we use impedance normalization and choose the positive sign in both $K(S)$ and $H(S)$, Eq. (6-65) gives

$$y_{11} = \frac{H_e + K_e}{H_o - K_o} = \frac{8.141552S^4 + 9.98938S^2 + 1.630867}{3.87868S^3 + 3.02304S}$$

Table 12-4 Natural modes for Chebyshev filters with $\alpha_p = 0.5$ dB and $\alpha_p = 1$ dB†

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
					$\frac{1}{2}$-dB ripple				
−2.8627752	−0.7128122 ±j1.0040425	−0.6264565	−0.1753531 ±j1.0162529	−0.3623196	−0.0776501 ±j1.0084608	−0.2561700	−0.0436201 ±j1.0050021	−0.1984053	−0.0278994 ±j1.0032732
		−0.3132282 ±j1.0219275	−0.4233398 ±j0.4209457	−0.1119629 ±j1.0115574	−0.2121440 ±j0.7382446	−0.0570032 ±j1.006405	−0.1242195 ±j0.8519996	−0.0344527 ±j1.0040040	−0.0809672 ±j0.9050658
				−0.2931227 ±j0.6251768	−0.2897940 ±j0.2702162	−0.1597194 ±j0.8070770	−0.1859076 ±j0.5692879	−0.0992026 ±j0.8829063	−0.1261094 ±j0.7182643
						−0.2308012 ±j0.4478939	−0.2192929 ±j0.1999073	−0.1519873 ±j0.6553170	−0.1589072 ±j0.4611541
								−0.1864400 ±j0.3486869	−0.1761499 ±j0.1589029
					1-dB ripple				
−1.9652267	−0.5488672 ±j0.8951286	−0.4941706	−0.1395360 ±j0.9833792	−0.2894933	−0.0621810 ±j0.9934115	−0.2054141	−0.0350082 ±j0.9964513	−0.1593305	−0.0224144 ±j0.9977755
		−0.2470853 ±j0.9659987	−0.3368697 ±j0.4073290	−0.0894584 ±j0.9901071	−0.1698817 ±j0.7272275	−0.0457089 ±j0.9952839	−0.0996950 ±j0.8447506	−0.0276674 ±j0.9972297	−0.1013166 ±j0.7143284
				−0.2342050 ±j0.6119198	−0.2320627 ±j0.2661837	−0.1280736 ±j0.7981557	−0.1492041 ±j0.5644443	−0.0796652 ±j0.8769490	−0.0650493 ±j0.9001063
						−0.1850717 ±j0.4429430	−0.1759983 ±j0.1982065	−0.1220542 ±j0.6508954	−0.1276664 ±j0.4586271
								−0.1497217 ±j0.3463342	−0.1415193 ±j0.1580321

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Table 12-5 Coefficients of the natural-mode polynomial for Chebyshev filters with $\alpha_p = 0.5$ dB and $\alpha_p = 1$ dB†

All coefficients are divided by $2^{n-1}k_p$. Hence $H(S) = 2^{n-1}k_p\left(S^n + \sum_{k=0}^{n-1} a_k S^k\right)$, where the a_k are the coefficients in the table.

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
					½-dB ripple					
1	2.8627752									
2	1.5162026	1.4256245								
3	0.7156938	1.5348954	1.2529130							
4	0.3790506	1.0254553	1.7168662	1.1973856						
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909					
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761				
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176			
8	0.0236907	0.1525444	0.5735604	1.1485894	2.1840154	2.1492173	2.6567498	1.1460801		
9	0.0111827	0.0941198	0.3408193	0.9836199	1.6113880	2.7814990	2.4293297	2.9027337	1.1425705	
10	0.0059227	0.0492855	0.2372688	0.6269689	1.5274307	2.1442372	3.4409268	2.7097415	3.1498757	1.1400664
					1-dB ripple					
1	1.9652267									
2	1.1025103	1.0977343								
3	0.4913067	1.2384092	0.9883412							
4	0.2756276	0.7426194	1.4539248	0.9528114						
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201					
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510				
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228			
8	0.0172267	0.1073447	0.4478257	0.8468243	1.8369024	1.6551557	2.4230264	0.9198113		
9	0.0076767	0.0706048	0.2441864	0.7863109	1.2016071	2.3781188	1.8814798	2.6709468	0.9175476	
10	0.0043067	0.0344971	0.1824512	0.4553892	1.2444914	1.6129856	2.9815094	2.1078524	2.9194657	0.9159320

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Table 12-6 Element values for Chebyshev filters with $\alpha_p = 0.5$ dB†

The transformer ratio t is prescribed for each circuit

Value of n	c_1 or l'_1	l_2 or c'_2	c_3 or l'_3	l_4 or c'_4	c_5 or l'_5	l_6 or c'_6	c_7 or l'_7	l_8 or c'_8	c_9 or l'_9	l_{10} or c'_{10}
					$t^2 = 3$					
1	1.3972									
2	2.8282	0.3109								
3	4.3200	0.4405	2.9371							
4	3.6172	0.6399	4.1985	0.3620						
5	4.7896	0.5293	5.8898	0.4809	3.1130					
6	3.7922	0.6851	4.8770	0.6852	4.3536	0.3722				
7	4.9305	0.5495	6.2770	0.5603	6.0535	0.4901	3.1632			
8	3.8560	0.6990	5.0230	0.7235	4.9937	0.6953	4.4022	0.3759		
9	4.9901	0.5572	6.3947	0.5770	6.4061	0.5671	6.1064	0.4936	3.1841	
10	3.8860	0.7051	5.0780	0.7348	5.1229	0.7314	5.0307	0.6993	4.4237	0.3776

$t^2 = 2$

1	1.0479									
2	1.5132	0.6538								
3	2.9431	0.6503	2.1903							
4	1.8158	1.1328	2.4881	0.7732						
5	3.2228	0.7645	4.1228	0.7116	2.3197					
6	1.8786	1.1884	2.7589	1.2403	2.5976	0.7976				
7	3.3055	0.7899	4.3575	0.8132	4.2419	0.7252	2.3566			
8	1.9012	1.2053	2.8152	1.2864	2.8479	1.2628	2.6310	0.8063		
9	3.3403	0.7995	4.4283	0.8341	4.4546	0.8235	4.2795	0.7304	2.3719	
10	1.9117	1.2127	2.8366	1.2999	2.8964	1.3054	2.8744	1.2714	2.6456	0.8104

$t^2 = 1$

1	0.6986									
2										
3	1.5963	1.0967	1.5963							
4										
5	1.7058	1.2296	2.5408	1.2296	1.7058					
6										
7	1.7373	1.2582	2.6383	1.3443	2.6383	1.2582	1.7373			
8										
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	
10										

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Table 12-7 Element values for Chebyshev filters with $\alpha_p = 1$ dB†

The transformer ratio t is prescribed for each circuit

Value of n	c_1 or l'_1	l_2 or c'_2	c_3 or l'_3	l_4 or c'_4	c_5 or l'_5	l_6 or c'_6	c_7 or l'_7	l_8 or c'_8	c_9 or l'_9	l_{10} or c'_{10}
					$t^2 = 3$					
1	2.0354									
2	2.5721	0.4702								
3	4.9893	0.4286	3.8075							
4	3.0355	0.7929	3.7589	0.5347						
5	5.3830	0.4915	6.6673	0.4622	3.9944					
6	3.1307	0.8287	4.1451	0.8467	3.8812	0.5475				
7	5.4978	0.5050	6.9839	0.5177	6.8280	0.4696	4.0473			
8	3.1647	0.8395	4.2237	0.8764	4.2404	0.8580	3.9186	0.5520		
9	5.5459	0.5101	7.0783	0.5288	7.1141	0.5232	6.8785	0.4724	4.0693	
10	3.1806	0.8442	4.2532	0.8851	4.3088	0.8857	4.2691	0.8623	3.9349	0.5541

$t^2 = 2$

1	1.5265									
2										
3	3.4774	0.6153	2.8540							
4										
5	3.7211	0.6949	4.7448	0.6650	2.9936					
6										
7	3.7916	0.7118	4.9425	0.7348	4.8636	0.6757	3.0331			
8										
9	3.8210	0.7182	5.0013	0.7485	5.0412	0.7429	4.9004	0.6797	3.0495	
10										

$t^2 = 1$

1	1.0177									
2										
3	2.0236	0.9941	2.0236							
4										
5	2.1349	1.0911	3.0009	1.0911	2.1349					
6										
7	2.1666	1.1115	3.0936	1.1735	3.0936	1.1115	2.1666			
8										
9	2.1797	1.1192	3.1214	1.1897	3.1746	1.1897	3.1214	1.1192	2.1797	
10										

† Reproduced by permission from L. Weinberg, "Network Synthesis and Analysis," McGraw-Hill, New York, 1962; reprinted by Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1975.

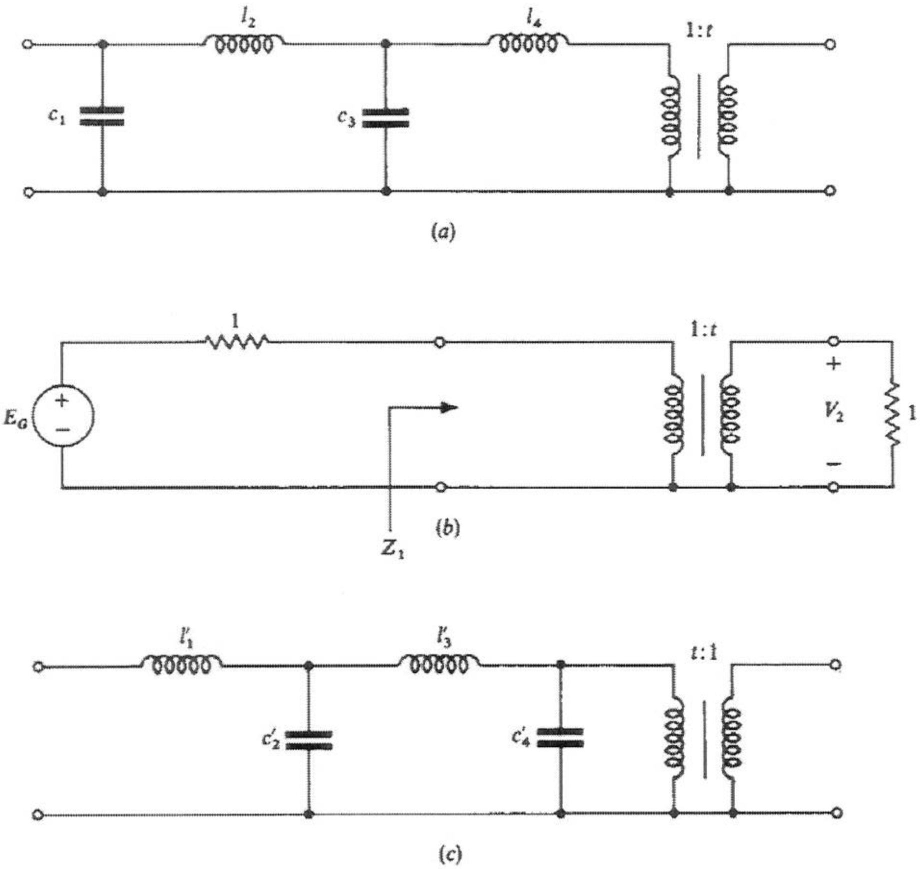


Figure 12-13 (*a*) Chebyshev filter; (*b*) equivalent circuit for $\omega = 0$; (*c*) dual filter.

Developing y_{11} into a ladder gives the normalized circuit of Fig. 12-13*a*. The element values can be found the usual way to be $c_1 = 2.09905$, $l_2 = 1.06444$, $c_3 = 2.831$, and $l_4 = 0.7892$. The transformer ratio can be found, for example, by developing y_{22}; it is much simpler to note, however, that at zero frequency, by Fig. 12-13*b*, the input impedance is $Z_1(0) = t^{-2}$. Hence, by (6-10), at zero frequency

$$\rho_1(0) = \frac{R_G - Z_1(0)}{R_G + Z_1(0)} = \frac{1 - t^{-2}}{1 + t^{-2}} = \frac{t^2 - 1}{t^2 + 1}$$

Also, by (6-21),

$$\rho_1(0) = \frac{K(0)}{H(0)} = \frac{0.508847}{1.12202} \approx 0.453510$$

Equating the two expressions and solving for t gives $t = \pm 1.630864$ (the negative sign means an inverting transformer).

Using the negative sign in $K(S)$ merely replaces the circuit by its dual. This is shown in Fig. 12-13*c*. The elements are $l'_1 = c_1$, $c'_2 = l_2$, $l'_3 = c_2$, and $c'_4 = l_4$.

As Eq. (12-50) shows, the natural modes S_k of the Chebyshev filter depend on k_p and hence on α_p. Therefore, any tabulation of the S_k, and hence of $H(S)$ and the

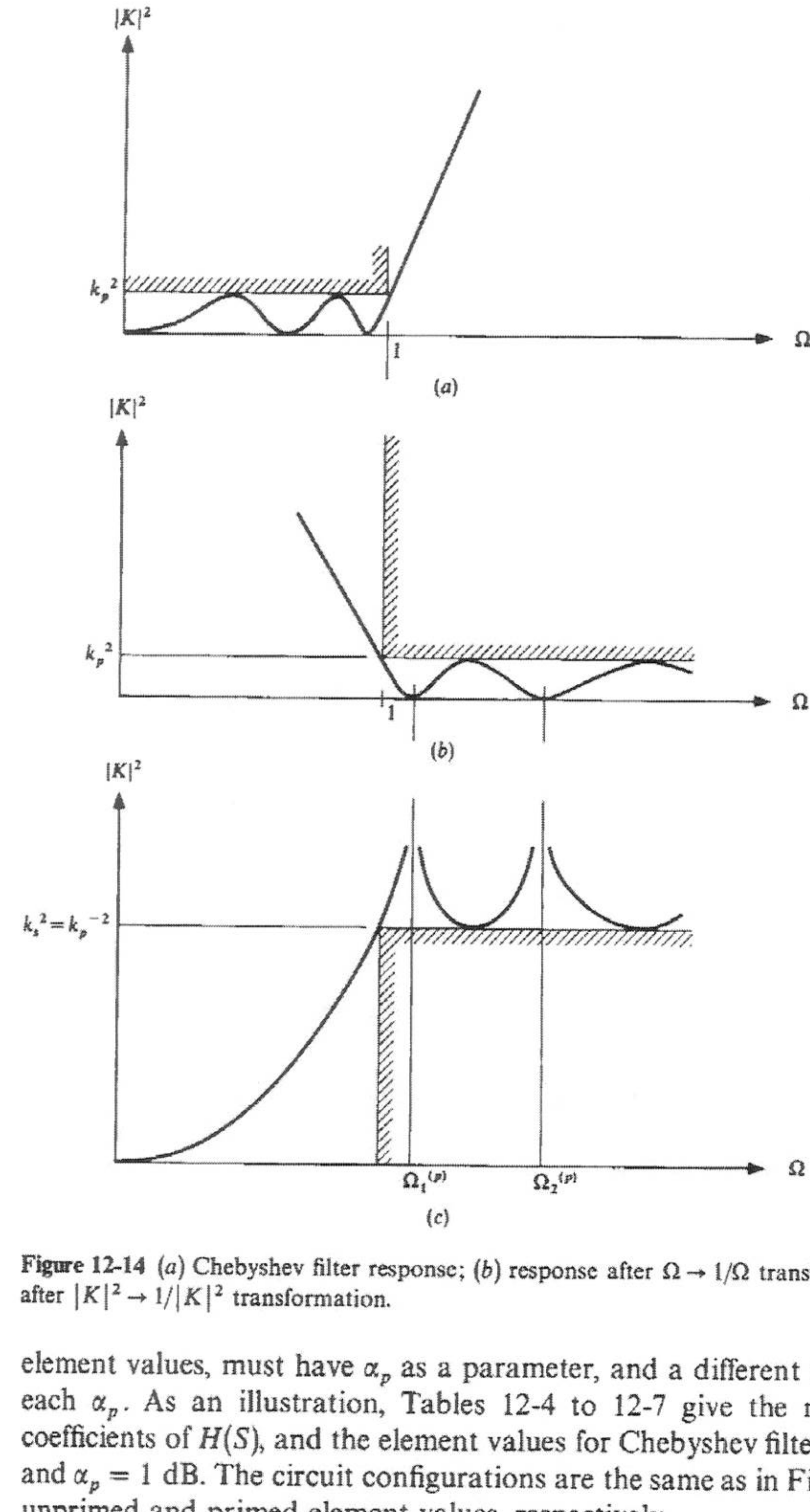


Figure 12-14 (*a*) Chebyshev filter response; (*b*) response after $\Omega \to 1/\Omega$ transformation; (*c*) response after $|K|^2 \to 1/|K|^2$ transformation.

element values, must have α_p as a parameter, and a different table will apply for each α_p. As an illustration, Tables 12-4 to 12-7 give the natural modes, the coefficients of $H(S)$, and the element values for Chebyshev filters with $\alpha_p = 0.5$ dB and $\alpha_p = 1$ dB. The circuit configurations are the same as in Fig. 12-13*a* and *c*, for unprimed and primed element values, respectively.

A filter response closely related to the Chebyshev response just discussed is the *inverse Chebyshev* (or *Chebyshev stopband*) characteristics. This can be obtained from the Chebyshev function in the following steps (Fig. 12-14):

1. Replace Ω by Ω^{-1} in Eq. (12-39). This will turn the $|K|^2$-vs.-Ω curve around, with the $\Omega = 1$ point as the pivot (Fig. 12-14*b*).
2. Replace $|K|^2$ by $|K|^{-2}$; this will result in the equal-ripple stopband characteristics shown in Fig. 12-14*c*.

Accordingly, the squared modulus of the characteristic function for the inverse Chebyshev function will be given by the expressions

$$|K|^2 = \frac{k_s^2}{\cos^2 nu(\Omega)} \tag{12-72}$$

and

$$u(\Omega) = \cos^{-1}\frac{1}{\Omega} \tag{12-73}$$

where

$$k_s^2 = 10^{\alpha_s/10} - 1 \triangleq \frac{1}{k_p^2} \tag{12-74}$$

From the steps leading to Eqs. (12-72) to (12-74), it is evident that the unit frequency is now the *stopband* limit frequency (Fig. 12-14). It can also be shown that Eq. (12-71), giving n in terms of k and k_1, remains valid for inverse Chebyshev filters. The proof is left as an exercise (Prob. 12-13). By the construction of $|K|^2$ it also follows that the loss poles are located at the reciprocal frequencies of the Chebyshev-filter reflection zeros. Hence, by (12-49),

$$\Omega_k^{(p)} = \frac{1}{\cos[(2k-1)/2\pi n]} \qquad k = 1, 2, \ldots, n \tag{12-75}$$

(Fig. 12-14*c*). The calculation of the natural modes is left to the reader (Prob. 12-15).

12-4 EQUAL-RIPPLE PASSBAND AND GENERAL STOPBAND FILTERS

A generalization of Chebyshev filters, important in practical problems, is provided by the class of filters with equal-ripple passbands and prescribed finite loss poles. A typical response is shown in Fig. 12-15*a*. These circuits are also often called (somewhat imprecisely) *general-parameter filters.*

The calculation of such a response with prescribed values of k_p (or, equivalently, α_p) and of the loss poles $\Omega_1, \Omega_2, \ldots$ requires some manipulations. Consider the transformation

$$Z \triangleq \sqrt{1 + S^{-2}} \qquad \text{Re } Z \geq 0 \tag{12-76}$$

Since in the passband $S = j\Omega$, $|\Omega| \leq 1$, the values of S in the passband transform into imaginary values jY of $Z = X + jY$. In the stopband $S = j\Omega$, $|\Omega| > 1$,

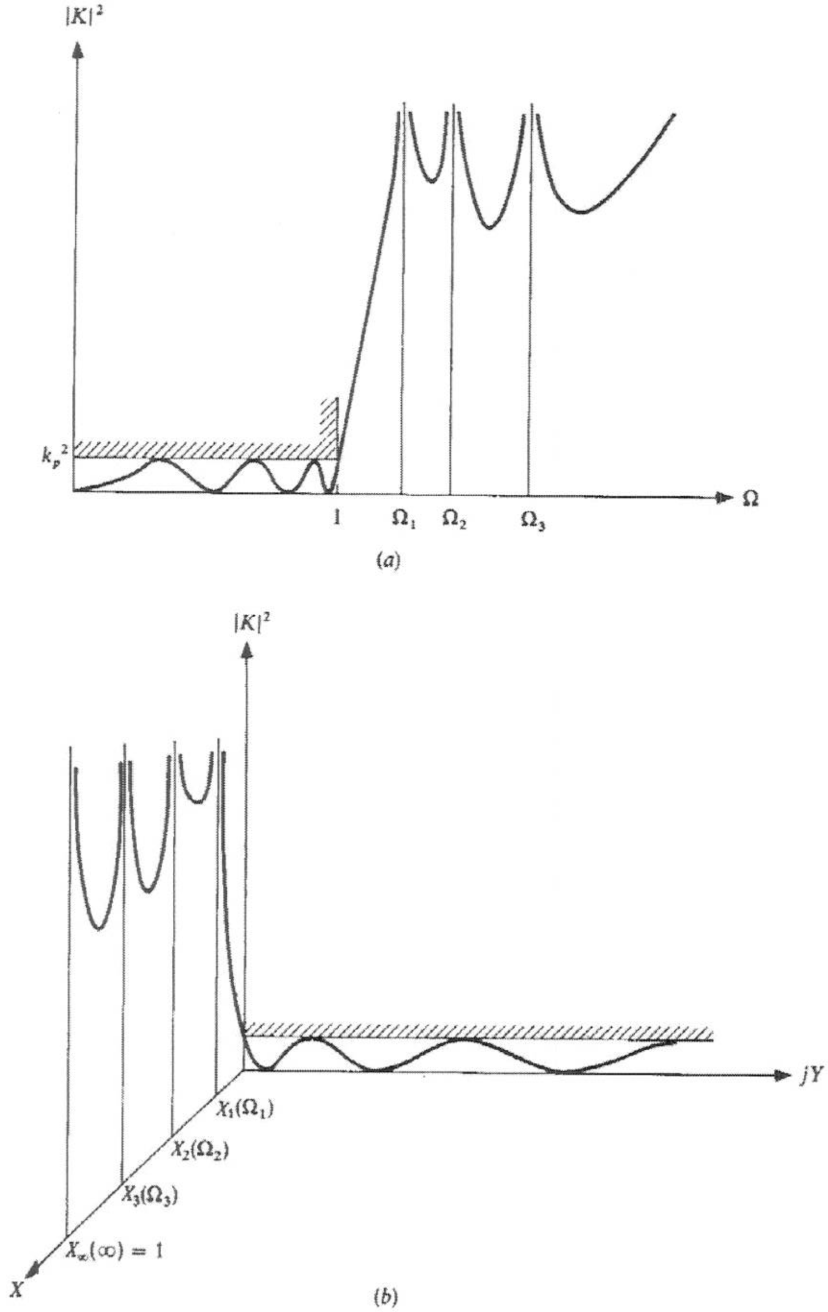


Figure 12-15 (*a*) General-parameter filter response; (*b*) filter response in the Z domain.

and hence Z is now real; that is, $Z = X$, where X, by (12-76), is nonnegative. The transformation is illustrated in Fig. 12-15*b* by showing the characteristic of Fig. 12-15*a* in the Z plane. A loss pole at $\Omega_i > 1$ is transformed to the real value

$$X_i = +\sqrt{1 - \Omega_i^{-2}} \tag{12-77}$$

and hence clearly $0 < X_i < 1$. Any loss pole at $\Omega \to \infty$ is transformed to $X_\infty = 1$.

$$\Omega = \frac{\omega}{\omega_p}$$

$$\Omega_i \to X_i = \sqrt{1-\Omega_i^{-2}}$$

$$\Omega_\infty \to X_\infty = 1$$

<u>Process</u>:

Form $P(Z) = \prod_{i=1}^{n} (Z + X_i)$

$$|K(Z)|^2 = \frac{k_p^2 Ev(P)}{[Ev(P)]^2 - [Od(P)]^2}$$

$Z^2 \to 1 + s^2$ gives $|K(s)|^2 = K(s)K(-s)$

<u>Result</u>

$$|K|^2 = k_p^2 \cos^2\left[\sum_i u_i(\Omega)\right]$$

$$u_i(\Omega) = \cos^{-1}\left(\Omega\sqrt{\frac{\Omega_i^2 - 1}{\Omega_i^2 - \Omega^2}}\right) = \tan^{-1}\frac{Z}{jX_i}$$

Equiripple for $\Omega \le 1$, real rational has poles Ω_i.

Next we define the polynomial

$$P(Z) \triangleq \prod_{i=1}^{n} (Z + X_i) \tag{12-78}$$

All desired loss poles X_i (including the X_∞) must be included in $P(Z)$; since poles at $+j\Omega_i$ and $-j\Omega_i$ map to the same X_i, such finite poles contribute a squared factor $(Z + X_i)^2$.

Let us now analyze the properties of the function

$$|K|^2 = k_p^2 \frac{[P_e(Z)]^2}{P(Z)P(-Z)} = k_p^2 \frac{[P_e(Z)]^2}{[P_e(Z)]^2 - [P_o(Z)]^2} = \frac{k_p^2}{1 - [P_o(Z)/P_e(Z)]^2} \tag{12-79}$$

Here, as before, P_e denotes the even part and P_o the odd part of $P(Z)$. Clearly, $|K|^2$ is an even rational function in Z, that is, a rational function of Z^2. By (12-76) therefore it is a rational function of S^2. Since $P(-Z) = \prod_{i=1}^{n} (-Z + X_i)$ is contained in the denominator, $|K|^2$ possesses the desired poles X_i. Finally, in terms of the transformed variable Z, $P(Z)$ is by (12-78) a strictly Hurwitz polynomial since all $X_i > 0$. Hence, $P_o(Z)/P_e(Z)$ is a reactance function in Z. Therefore, for $Z = jY$, that is, for values of Z in the transformed passband, the function $P_o(jY)/P_e(jY)$ is pure imaginary and acts as a reactance (Fig. 12-16*a*). Hence, $-(P_o/P_e)^2$ is a nonnegative function which oscillates between 0 and ∞ as Y increases from 0 to ∞. Now, as the last expression in (12-79) shows, $|K|^2 \to 0$ when $-(P_o/P_e)^2 \to \infty$; $|K|^2 = k_p^2$ when $-(P_o/P_e)^2 = 0$; and $0 < |K|^2 < k_p^2$ when $0 < -(P_o/P_e)^2 < \infty$. Thus, $|K|^2$ oscillates between 0 and k_p^2 as Y increases from 0 to ∞.

The above results show that $|K|^2$ as given in (12-79) does indeed have the desired response illustrated in Figs. 12-14 and 12-15 and is also a realizable function.†

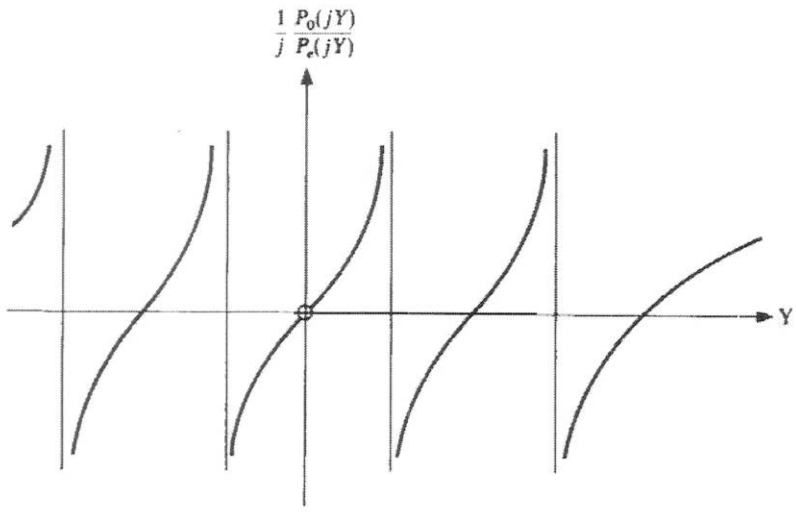
It is possible to bring (12-79) to a form which indicates clearly that it is an extension of the Chebyshev filter function defined in Eqs. (12-39) and (12-40). It is easy to see that

$$|K|^2 = \frac{k_p^2}{4} \frac{[P(Z) + P(-Z)]^2}{P(Z)P(-Z)} = \frac{k_p^2}{2}\left[1 + \frac{1}{2}\frac{P(Z)}{P(-Z)} + \frac{1}{2}\frac{P(-Z)}{P(Z)}\right] \tag{12-80}$$

Let us now define

$$u_i \triangleq \tan^{-1} \frac{Z}{jX_i} \tag{12-81}$$

† See also Prob. 12-17, which proves that $|K|^2 > 0$ in the stopband as well as in the passband.



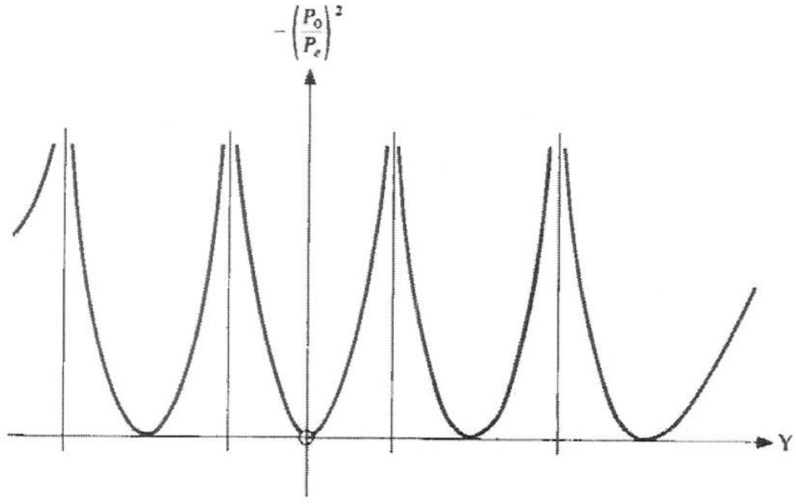


Figure 12-16 (*a*) The response of the quasi reactance P_o/P_e; (*b*) the behavior of $-(P_o/P_e)^2$ for $Z = jY$.

Then
$$\frac{Z + X_i}{-Z + X_i} = \frac{1 + j \tan u_i}{1 - j \tan u_i} = \frac{\cos u_i + j \sin u_i}{\cos u_i - j \sin u_i} = e^{j2u_i} \qquad (12\text{-}82)$$

Hence, by (12-78),

$$\frac{P(Z)}{P(-Z)} = \prod_{i=1}^{n} \frac{Z + X_i}{-Z + X_i} = \exp\left(j2\sum_{i=1}^{n} u_i\right) \qquad (12\text{-}83)$$

and (12-80) gives

$$|K|^2 = \frac{k_p^2}{2}\left[1 + \frac{\exp\left(j2\sum_{i=1}^{n} u_i\right) + \exp\left(-j2\sum_{i=1}^{n} u_i\right)}{2}\right]$$
$$= \frac{k_p^2}{2}\left[1 + \cos\left(2\sum_{i=1}^{n} u_i\right)\right] = k_p^2 \cos^2\left(\sum_{i=1}^{n} u_i\right) \qquad (12\text{-}84)$$

Equation (12-84), together with the defining equation† of the u_i

$$u_i \triangleq \tan^{-1}\frac{Z}{jX_i} = \cos^{-1}\left\{\left[1 - \left(\frac{Z}{X_i}\right)^2\right]^{-1/2}\right\} = \cos^{-1}\left(\Omega\sqrt{\frac{\Omega_i^2 - 1}{\Omega_i^2 - \Omega^2}}\right) \qquad (12\text{-}85)$$

is compared in (12-86) with the basic equations (12-39) and (12-40) of Chebyshev filters:
Chebyshev filters:

$$|K|^2 = k_p^2 \cos^2 nu(\Omega) \qquad \text{where } u(\Omega) \triangleq \cos^{-1}\Omega \qquad (12\text{-}86a)$$

General-parameter filters:

$$|K|^2 = k_p^2 \cos^2\left[\sum_{i=1}^{n} u_i(\Omega)\right] \qquad \text{where } u_i(\Omega) \triangleq \cos^{-1}\left(\Omega\sqrt{\frac{\Omega_i^2 - 1}{\Omega_i^2 - \Omega^2}}\right) \qquad (12\text{-}86b)$$

It is clear that if all $\Omega_i \to \infty$, then $u_i(\Omega) \to \cos^{-1}\Omega$ for all i and $\sum_{i=1}^{n} u_i(\Omega) \to nu(\Omega)$. Hence, the Chebyshev filter is a special case of the general-parameter one.

Equation (12-86) can be also regarded as the defining equation of a general-parameter filter, and all properties of $|K|^2$ can be derived from it (see, for example, Probs. 12-19 to 12-21).

The design procedure to be used when the parameters $\alpha_p, \Omega_1, \Omega_2, \ldots, \Omega_n$ are prescribed can be readily found from Eqs. (12-76) to (12-79). The steps are the following:

1. Calculate from (12-69)

$$k_p^2 = 10^{\alpha_p/10} - 1$$

and from (12-77)

$$X_i = +\sqrt{1 - \Omega_i^{-2}} \qquad i = 1, 2, \ldots, n$$

† See Prob. 12-18 for the detailed derivation of Eq. (12-85).

Note that if there are n_∞ loss poles specified at $\Omega \to \infty$, n_∞ of the X_i will equal 1. Note also that a finite pole pair at $\pm j\Omega_i$ will result in two equal X_i values.

2. Calculate the coefficients of the polynomial

$$P(Z) = \prod_{i=1}^{n} (Z + X_i) = (Z + 1)^{n_\infty} \prod_{i=1}^{(n-n_\infty)/2} (Z + X_i)^2 \qquad (12\text{-}87)$$

Here, on the right-hand side, the loss poles at infinity have been separated and collected in the first factor. Each of the remaining factors corresponds to a $\pm j\Omega_i$ pole pair; hence each X_i enters only one (squared) factor.

3. From (12-79), calculate the coefficients of the rational function

$$|K|^2 = k_p^2 \frac{\left\{ \text{Ev} \left[(Z + 1)^{n_\infty} \prod_{i=1}^{(n-n_\infty)/2} (Z + X_i)^2 \right] \right\}^2}{(1 - Z^2)^{n_\infty} \prod_{i=1}^{(n-n_\infty)/2} (X_i^2 - Z^2)^2} \qquad (12\text{-}88)$$

where Ev stands for "even part of." Only even powers of Z enter $|K|^2$.

4. Using (12-76), replace Z^2 by $1 + S^{-2}$. The result is a rational function of S^2.
5. Calculate $K(S)$ and $H(S)$ in the usual manner and complete the synthesis.

Example 12-5 Find $K(S)$ and $H(S)$ and design the filter from the following specifications:

1. $\alpha \le 0.28$ dB for $|f| \le 10$ kHz.
2. Desired loss poles: $f_1 = 26$ kHz and one pole at infinity.
3. Both terminations are to be 100 Ω.

Using $\omega_0 = 2\pi f_p = 2\pi 10^4$ rad/s as the unit radian frequency, we get $\Omega_1 = 2.6$, $\Omega_2 = -2.6$, and $\Omega_3 \to \infty$. Hence, by (12-77), $X_1 = X_2 = \sqrt{1 - \Omega_1^{-2}} \approx 0.923077$. Also, $X_3 = 1$. From (12-69),

$$k_p^2 = 10^{\alpha_p/10} - 1 = 10^{0.028} - 1 \approx 0.0665961$$

Next, from (12-87),

$$P(Z) = (Z + 1)(Z + X_1)^2 = Z^3 + a_2 Z^2 + a_1 Z + a_0$$

where $a_2 = 2.846154$, $a_1 = 2.698225$, and $a_0 = 0.852071$. Substituting into (12-88), we get

$$|K|^2 = \frac{k_p^2 (a_2 Z^2 + a_0)^2}{(1 - Z^2)(X_1^2 - Z^2)^2}$$

Next, replacing Z^2 by $1 + 1/S^2$, we obtain

$$|K|^2 = \frac{-(b_3 S^3 + b_1 S)^2}{(S^2 + \Omega_1^2)^2}$$

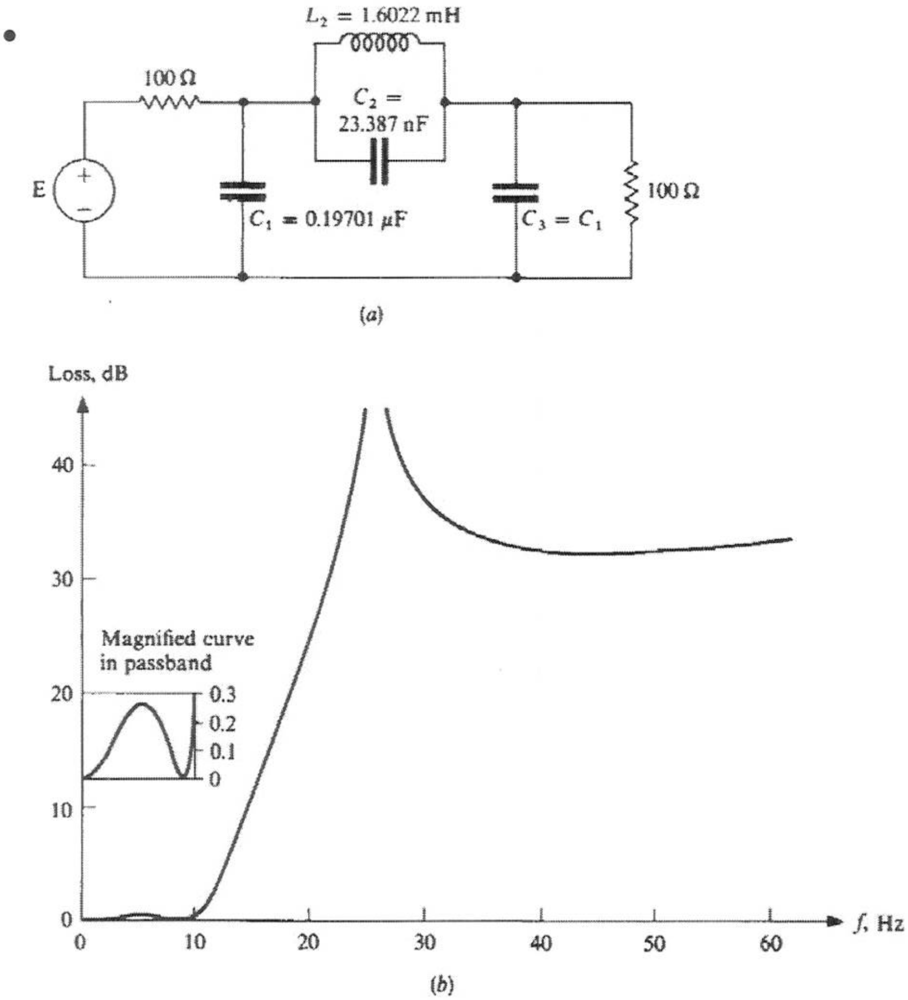


Figure 12-17 (*a*) General-parameter filter; (*b*) its loss response.

where $b_3 = 6.451555$ and $b_1 = 4.965117$. Since

$$|K|^2 = K(S)K(-S) = \frac{(b_3 S^3 + b_1 S)(-b_3 S^3 - b_1 S)}{(S^2 + \Omega_1^2)(S^2 + \Omega_1^2)}$$

clearly we can choose

$$K(S) = \pm \frac{b_3 S^3 + b_1 S}{S^2 + \Omega_1^2}$$

By the Feldtkeller equation (6-42), $E(S)E(-S) = -(b_3 S^3 + b_1 S)^2 + (S^2 + \Omega_1^2)^2$, and the Hurwitz factor is

$$E(S) = 6.451555S^3 + 9.423913S^2 + 11.77046S + 6.76$$

Hence, from Table 6-1, choosing the plus sign for $K(S)$, we obtain

$$z_{11} = z_{22} = R_1 \frac{E_e}{E_o + F_o} = \frac{942.3913S^2 + 676}{12.90311S^3 + 16.735637S}$$

1 pole at ∞

2 poles at 26kHz

$$f_p = 10kHz$$

$$P(Z) = (Z+1)(Z+X_1)$$

$$X_{1,2} = +\sqrt{1-\left(\frac{10}{26}\right)^2} \approx 0.923$$

$$Ev(P) = \ldots$$

$$Od(P) = \ldots$$

$$|K(Z^2)|^2 \rightarrow K(s)K(-s)$$

$$H(s)H(-s) = K(s)K(-s) + 1 \rightarrow H(s), K(s)$$

$$E(s)E(-s) = F(s)F(-s) + P(s)P(-s) \rightarrow 0$$

Synthesizing the two-port using ladder development gives the circuit of Fig. 12-17*a*. Its loss response is shown in Fig. 12-17*b*.

In practical filter design, it is often the minimum permissible value of the loss in the stopband which is specified, rather than the location of the loss poles. Then an iterative design procedure, based on Eq. (12-80), can be used. This procedure is beyond the scope of our book; the reader should consult Refs. 2 and 3.

Another practical aspect concerns the numerical accuracy of filter design calculations. It can be shown[2,3] that the use of the transformed variable Z defined in (12-76) is very advantageous in that it preserves the accuracy of the calculations even for filters of very high order. This is only true, however, if *all* calculations from the construction of $|K|^2$ on all the way to the calculation of the element values are done in terms of Z rather than S. Again, the reader should consult Refs. 2 and 3 for the details of this process.

A special case of the class of general-parameter filters is obtained when the loss poles are located in such a manner that the stopband as well as the passband is equal-ripple (Fig. 12-18). Since these filters can be treated purely analytically and their $K(S)$ constructed in terms of elliptic functions, they are called *elliptic filters*. A detailed analysis[5] is quite lengthy and is therefore not included here. An important property of elliptic filters is that for given α_p, α_s, f_p, and f_s they require the lowest possible degree of all lumped linear filters. They are thus very economical. Since their design is complicated, industrial filter designers often rely on the many excellent reference works[1,6–8] listing the zeros, poles, and element values of elliptic filters. Since the actual values depend on α_p, α_s, k, and n, these tables tend to be voluminous. Table 12-8 contains a tabulation of the natural modes, loss poles, and element values of elliptic filters with $n = 3$, $\Omega_p = 1$, $R_G = R_L = 1\ \Omega$, and a maximum passband reflection factor of 20 percent. The first column contains the values of $\theta \triangleq \sin^{-1} k$; the second Ω_s; the third $A_{\min} \equiv \alpha_s$; the next four the zeros and poles, and the last three the element values.

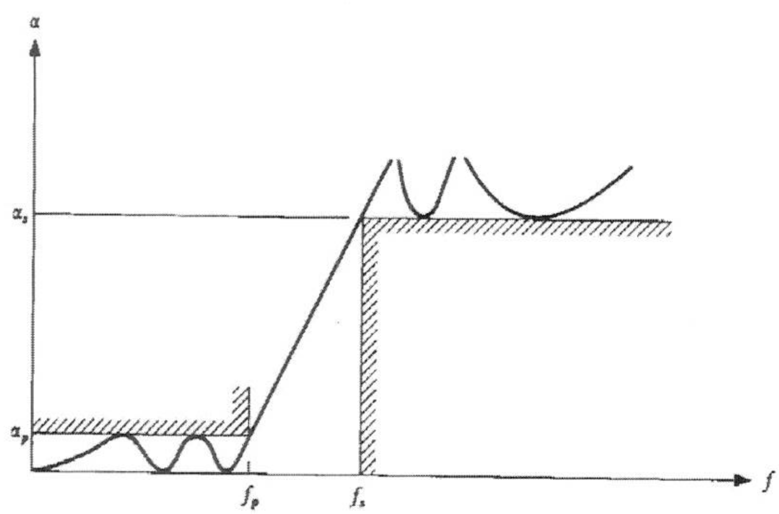
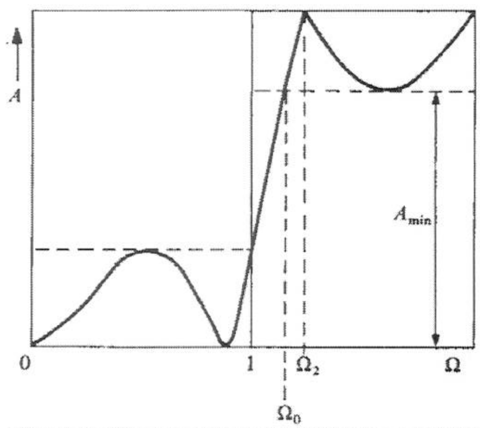
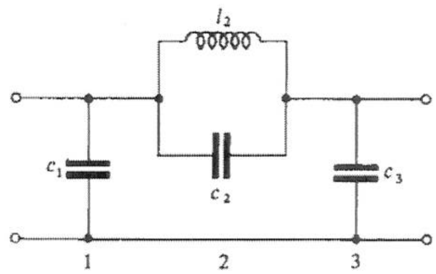


Figure 12-18 Elliptic filter response.

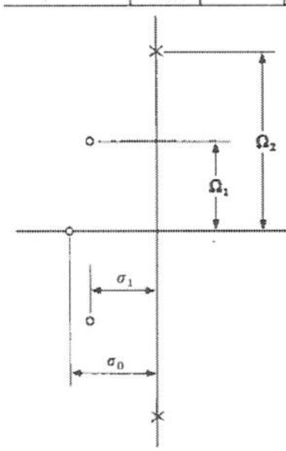
Table 12-8 Poles, zeros, and element values for third-order elliptic filters with 20 percent passband reflection coefficient, $\Omega_p = 1$, $n = 3$, and equal terminations of 1 Ω†

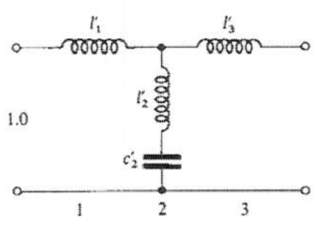




θ	Ω_s	A_{min}, dB	σ_0	σ_1	Ω_1	Ω_2	$c_1 = c_3$	c_2	l_2
1	57.2987	115.77	0.84082	0.42027	1.13143	66.1616	1.1893	0.0002	1.1540
2	28.6537	97.70	0.84114	0.42001	1.13150	33.0839	1.1889	0.0008	1.1533
3	19.1073	87.13	0.84169	0.41958	1.13160	22.0595	1.1881	0.0018	1.1522
4	14.3356	79.63	0.84246	0.41899	1.13175	16.5483	1.1870	0.0032	1.1507
5	11.4737	73.81	0.84345	0.41822	1.13194	13.2424	1.1856	0.0050	1.1488
6	9.5668	69.05	0.84466	0.41728	1.13218	11.0392	1.1839	0.0072	1.1464
7	8.2055	65.03	0.84609	0.41617	1.13245	9.4661	1.1819	0.0098	1.1436
8	7.1853	61.54	0.84776	0.41489	1.13276	8.2868	1.1796	0.0128	1.1404
9	6.3925	58.46	0.84965	0.41344	1.13311	7.3700	1.1770	0.0162	1.1367
10	5.7588	55.70	0.85177	0.41182	1.13350	6.6370	1.1740	0.0200	1.1326
11	5.2408	53.20	0.85413	0.41003	1.13392	6.0377	1.1708	0.0243	1.1281
12	4.8097	50.92	0.85673	0.40807	1.12438	5.5386	1.1672	0.0290	1.1231
13	4.4454	48.82	0.85957	0.40593	1.13466	5.1166	1.1634	0.0342	1.1177
14	4.1336	46.87	0.86266	0.40363	1.13538	4.7552	1.1592	0.0398	1.1119
15	3.8637	45.05	0.86600	0.40115	1.13592	4.4423	1.1547	0.0458	1.1057
16	3.6280	43.35	0.86959	0.39851	1.13649	4.1088	1.1500	0.0524	1.0990
17	3.4203	41.75	0.87345	0.39569	1.13709	3.9277	1.1449	0.0594	1.0919
18	3.2361	40.23	0.87759	0.39270	1.13770	3.7137	1.1595	0.0669	1.0844
19	3.0716	38.80	0.88199	0.38954	1.13833	3.5224	1.1338	0.0749	1.0764
20	2.9238	37.44	0.88668	0.38621	1.13897	3.3505	1.1278	0.0834	1.0681
21	2.7904	36.14	0.89167	0.38272	1.13963	3.1951	1.1215	0.0925	1.0593
22	2.6695	34.90	0.89695	0.37905	1.14029	3.0541	1.1149	0.1021	1.0500
23	2.5593	33.71	0.90254	0.37521	1.14096	2.9256	1.1080	0.1123	1.0404
24	2.4586	32.57	0.90845	0.37120	1.14162	2.8079	1.1008	0.1231	1.0303
25	2.3662	31.47	0.91469	0.36702	1.14228	2.6999	1.0933	0.1345	1.0199
26	2.2812	30.41	0.92127	0.36268	1.14294	2.6003	1.0855	0.1466	1.0090
27	2.2027	29.39	0.92820	0.35817	1.14358	2.5083	1.0773	0.1593	0.9976
28	2.1301	28.41	0.93550	0.35349	1.14420	2.4231	1.0682	0.1728	0.9850
29	2.0627	27.45	0.94318	0.34864	1.14480	2.3438	1.0602	0.1869	0.9733
30	2.0000	26.53	0.95125	0.34364	1.14538	2.2701	1.0512	0.2019	0.9612
θ	Ω_s	A_{min}, dB	σ_0	σ_1	Ω_1	Ω_2	$l'_1 = l'_3$	l'_2	c'_2

θ	Ω_s	$A_{min.}$ dB	σ_0	σ_1	Ω_1	Ω_2	$c_1 = c_3$	c_2	l_2
31	1.9416	25.63	0.95973	0.33847	1.14592	2.2012	1.0420	0.2176	0.9483
32	1.8871	24.76	0.96863	0.33313	1.14643	2.1368	1.0324	0.2343	0.9349
33	1.8361	23.92	0.97799	0.32764	1.14689	2.0765	1.0225	0.2518	0.9212
34	1.7883	23.09	0.98780	0.32199	1.14730	2.0199	1.0123	0.2702	0.9070
35	1.7434	22.29	0.99810	0.31619	1.14766	1.9666	1.0019	0.2897	0.8925
36	1.7013	21.51	1.00890	0.31023	1.14796	1.9165	0.9912	0.3103	0.8776
37	1.6616	20.74	1.02024	0.30412	1.14819	1.8602	0.9802	0.3320	0.8623
38	1.6243	20.00	1.03213	0.29786	1.14835	1.8245	0.9689	0.3549	0.8466
39	1.5890	19.27	1.04460	0.29147	1.14842	1.7823	0.9573	0.3791	0.8305
40	1.5557	18.56	1.05768	0.28493	1.14841	1.7423	0.9455	0.4047	0.8141
41	1.5243	17.86	1.07140	0.27825	1.14830	1.7044	0.9334	0.4318	0.7973
42	1.4945	17.18	1.08579	0.27145	1.14810	1.6684	0.9210	0.4605	0.7801
43	1.4663	16.52	1.10089	0.26452	1.14778	1.6343	0.9084	0.4909	0.7627
44	1.4396	15.86	1.11673	0.25747	1.14735	1.6018	0.8055	0.5232	0.7448
45	1.4142	15.22	1.13336	0.25031	1.14679	1.5710	0.8823	0.5576	0.7267
46	1.3902	14.60	1.15082	0.24304	1.14611	1.5415	0.8689	0.5942	0.7082
47	1.3673	13.98	1.16915	0.23567	1.14528	1.5135	0.8553	0.6331	0.6895
48	1.3456	13.38	1.18840	0.22821	1.14432	1.4868	0.8415	0.6747	0.6705
49	1.3250	12.79	1.20862	0.22067	1.14320	1.4613	0.8274	0.7192	0.6511
50	1.3054	12.22	1.22988	0.21306	1.14192	1.4369	0.8131	0.7668	0.6316
51	1.2868	11.65	1.25221	0.20539	1.14048	1.4137	0.7986	0.8179	0.6118
52	1.2690	11.10	1.27570	0.19766	1.13887	1.3914	0.7839	0.8728	0.5918
53	1.2521	10.56	1.30040	0.18990	1.13709	1.3702	0.7600	0.9319	0.5716
54	1.2361	10.03	1.32639	0.18211	1.13512	1.3498	0.7539	0.9958	0.5512
55	1.2208	9.51	1.35374	0.17431	1.13297	1.3303	0.7387	1.0648	0.5306
56	1.2062	9.01	1.38253	0.16652	1.13064	1.3117	0.7233	1.1397	0.5100
57	1.1924	8.51	1.41284	0.15873	1.12811	1.2938	0.7078	1.2210	0.4892
58	1.1792	8.03	1.44478	0.15098	1.12540	1.2767	0.6921	1.3097	0.4684
59	1.1666	7.57	1.47842	0.14328	1.12249	1.2603	0.6764	1.4065	0.4476
60	1.1547	7.11	1.51387	0.13565	1.11939	1.2446	0.6606	1.5127	0.4268
θ	Ω_s	$A_{min.}$ dB	σ_0	σ_1	Ω_1	Ω_2	$l'_1 = l'_3$	l'_2	c'_2





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